Multivariable Calculus

Quiz 6 **SOLUTIONS**

1) Consider the space curve with t restricted to $t > 0$.

$$\vec{r}(t) = \left\langle 2t^2,\ 3\sin(t) - 3t\cos(t),\ 3\cos(t) + 3t\sin(t) \right\rangle$$

a) Compute the unit tangent vector, $\hat{T}(t)$ for this space curve.

Solution:

$$\begin{aligned}\vec{r}\,'(t) &= \langle 4t,\ 3\cos(t) - 3\cos(t) + 3t\sin(t),\ -3\sin(t) + 3\sin(t) + 3t\cos(t)\rangle \\ &= \langle 4t,\ 3t\sin(t),\ 3t\cos(t)\rangle \\ &= t\,\langle 4,\ 3\sin(t),\ 3\cos(t)\rangle\end{aligned}$$

Notice that the speed of this parametrization is $\|\vec{r}\,'(t)\| = 5t$. So, the unit tangent vector is

$$\hat{T}(t) = \left\langle \frac{4}{5},\ \frac{3}{5}\sin(t),\ \frac{3}{5}\cos(t) \right\rangle.$$

b) Compute the unit normal vector, $\hat{N}(t)$ for this space curve.

Solution:

$$\begin{aligned}\hat{T}\,'(t) &= \left\langle 0,\ \frac{3}{5}\cos(t),\ -\frac{3}{5}\sin(t) \right\rangle \\ &= \frac{3}{5}\langle 0,\ \cos(t),\ -\sin(t)\rangle\end{aligned}$$

This means the unit normal vector is given by

$$\hat{N}(t) = \langle 0,\ \cos(t),\ -\sin(t)\rangle.$$

c) Compute the unit binormal vector, $\hat{B}(t)$ for this space curve.

Solution:

$$\hat{B}(t) = \hat{T}(t) \times \hat{N}(t) = \left\langle -\frac{3}{5},\ \frac{4}{5}\sin(t),\ \frac{4}{5}\cos(t) \right\rangle$$

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2) Consider the graph $y = \ln(x)$ for $x > 0$ which we can parametrize as

$$\vec{r}(x) = \langle x,\ \ln(x),\ 0\rangle.$$

Find the value of x for which the curvature of this graph is a maximum.

Solution: To compute the curvature, we the velocity and acceleration vectors for this curve.

$$\vec{r}\,'(x) = \left\langle 1,\ \frac{1}{x},\ 0\right\rangle$$
$$\vec{r}\,''(x) = \left\langle 0,\ -\frac{1}{x^2},\ 0\right\rangle$$

An easy calculation shows that

$$\|\vec{r}\,'(x) \times \vec{r}\,''(t)\| = \left\|\left\langle 0,\ 0,\ -\frac{1}{x^2}\right\rangle\right\| = \frac{1}{x^2}.$$

The speed of this curve is

$$\|\vec{r}\,'(x)\| = \sqrt{1 + \frac{1}{x^2}} = \frac{\sqrt{x^2+1}}{x}.$$

That means the curvature is given by

$$\kappa(x) = \frac{\|\vec{r}\,'(x) \times \vec{r}\,''(x)\|}{\|\vec{r}\,'(x)\|^3} = \frac{x}{(x^2+1)^{3/2}}.$$

To find the point of maximum curvature, we compute the derivative and set it equal to zero.

$$\kappa'(x) = \frac{1-2x^2}{(x^2+1)^{5/2}} = 0$$
$$x = \frac{\sqrt{2}}{2} \text{ (recall that } x > 0).$$

A quick check of the derivative shows that $\kappa(x)$ is increasing on $0 < x < \frac{\sqrt{2}}{2}$ and decreasing on $\frac{\sqrt{2}}{2} < x < \infty$. Hence, the point of maximum curvature on the graph $y = \ln(x)$ is at $x = \frac{\sqrt{2}}{2}$.